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During this period, the primary area of investigation was the study of the behavior of stochastic processes whose power spectra are described by power-law or piecewise power-law behavior. The attached paper (to be submitted to *Geophysical Research Letters*) gives the details of the analysis and the conclusions we have reached. We are extending this analysis to compare the detection capabilities of different measurement techniques (e.g., gravimetry and GPS for the vertical, and seismometers and GPS for the horizontal), both in general and for the specific case of the deformations produced by a dislocation in a half-space (which applies to seismic or preseismic sources).

If the source of deformation can be approximated by a dislocation in a halfspace, the average displacement $\bar{x}$ at a distance Δ from a source with moment M_0 is very nearly $\bar{x} = K_x M_0 / \Delta^2$, for distances of more than a few source dimensions. Similarly, the average displacement gradient (strain or tilt) $\bar{\varepsilon}$ can be approximated by $\bar{\varepsilon} = K_\varepsilon M_0 / \Delta^3$. (At close distances these expressions overestimate the effects). For vector horizontal displacement and maximum extension around a vertical strike-slip fault, we find $K_x \approx 5 \times 10^{-12}$ and $K_\varepsilon \approx 10^{-11}$ (Δ in m, M_0 in N-m).

Even though strains and tilts decay much faster with distance than displacements do, the much higher resolution with which strain can be observed over short times makes such observations considerably more sensitive to rapidly-changing sources. Suppose that over a time t we can resolve changes in strain of $\varepsilon(t)$ and in displacement of $x(t)$. Then we can, for example, detect strain changes from a dislocation that releases a moment $M_0(t)$ for distances less than Δ_ε, where $\varepsilon(t) = K_\varepsilon M_0 / \Delta_\varepsilon^3$. We can compute a similar distance Δ_x for displacement measurements. The ratio of areas within which the moment release is detectable then reflects the relative density of measurements needed to attain the same detection capability. This is

$$A(t, M_0) = \frac{\Delta_\varepsilon^2}{\Delta_x^2} = \frac{K_\varepsilon^{0.67}}{K_x} \frac{x(t)}{\varepsilon(t)^{0.67}} M_0^{-0.33}$$

Using the values of K_x and K_ε given above, and the short-term resolutions of 10^{-10} for strain ($\varepsilon(t)$) and 2×10^{-3} m for planned continuous GPS systems ($x(t)$), we find A to be about 100 for $M_0 = 10^{18}$ ($M_w = 6$); for such rapid changes, a single strain installation could be expected to cover the same area as 100 geodetic stations. The scaling with moment means that the area "covered" by strain measurements exceeds that for displacement measurements even for the largest earthquakes. For larger t this ratio of areas will not be as great, but for any t, the smaller the source, the greater the relative advantage of measuring displacement gradient rather than displacement. To take an example for a small event, the maximum surface displacement expected from a magnitude 4 earthquake at 10 km depth is only 15 microns, while its strain is an easily detectable 2×10^{-9}.

THE TIME-DOMAIN BEHAVIOR OF POWER-LAW NOISES

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Abstract The power spectra of many geophysical phenomena are well approximated by a power-law dependence on frequency or wavenumber. I derive a simple expression for the root-mean-square variability of a process with such a spectrum over an interval of time or space. The resulting expression yields the power-law time dependence characteristic of fractal processes, but can be generalized to give the temporal variability for more general spectral behaviors. The method is applied to spectra of crustal strain (to show what size of strain events can be detected over periods of months to seconds) and of sea level (to show the difficulty of extracting long-term rates from short records).

1. Introduction

Many types of geophysical data come from processes so complex that their outcome is best taken to be random, even if the underlying physics is not; the most efficient characterizations of such data are likely to be a statistical model. The aim of this paper is to develop a useful relation for a particular (but common) class of such models, and show several applications if it.

The particular type of model considered might be called the power-law process. This is a one-dimensional stochastic process whose behavior in the time domain (or space domain if appropriate) we denote by $x(t)$; the time-domain behavior is such that its power spectrum has the form

$$P_x(f) = P_0 \left[\frac{f}{f_0} \right]^\nu \qquad (1)$$

where f is spatial or temporal frequency, P_0 and f_0 are normalizing constants, and ν is the spectral index. The reason for adopting this form is that it is observed to be a good fit to the spectra of a wide variety of geophysical phenomena, often applying over many decades of frequency. The index ν generally falls in the range -3 to -1, meaning that the energy in low frequencies exceeds that at high frequencies (a "red" spectrum). Spectra of this form have been found for bathymetry with $\nu = -2.3$ [Malinaverno, 1989]; fault and joint geometry, with $\nu = -2$ [Power *et al.*, 1987]; and crustal deformation, with $\nu = -2.7$ [Wyatt *et al.*, 1988].

Despite the ubiquity of stochastic processes with power-law spectra, they have received relatively little attention in the statistical literature. The only exception has been the case $\nu = -2$, which corresponds to a random walk (Brownian motion): this is the integral of white noise (for which $\nu = 0$), the spectral index being shifted to -2 by the operation of integration (and squaring to get power). Mandelbrot and Van Ness

[1968] developed mathematical forms for processes that have power spectra close to (1), those with $-3 < v < -1$ being termed "fractional Brownian motions," and those with $-1 < v < 1$, "fractional Gaussian noises." Mandelbrot [1983] provides a general discussion of these, and Feder [1988] a readable introduction.

For clarity, it should be noted that the models introduced by Mandelbrot do not have exact power-law spectra [Graf 1983], and indeed are more usually discussed in terms of their behavior under changes in time scale (assuming them to represent a time series). In this view, the important parameter becomes a number H, called by Mandelbrot [1983] the Hurst exponent; as we will see below, $H = -\frac{1}{2}(v + 1)$. In practice, both Mandelbrot's models and the spectral form (1) are mathematical idealizations, and in different situations either one might be the better description of actual data. I have chosen to use (1) as a model because my main purpose is the interpretation of spectra, for which (1) can easily be generalized (Section 4). It is worth noting that recent statistical studies [Mohr, 1981; Graf, 1983] have shown that the best method for determining H is to compute the power spectrum, fit a function of the form (1) to it, and then find H from the v so determined; in that sense v could be regarded as the more fundamental parameter.

However, it should also be noted that (1) is not in general a complete specification of a process; the power spectrum is only a summary of second moments (variance versus frequency). Stochastic processes with identical power spectra can have very different appearances in the time domain [Press, 1978]. We address this point more fully below.

2. Time-Domain Variation

The question to be addressed here is how to go from the power spectrum (1) to the variation of the process over time T; that is, to the statistics of

$$y_T(t) = x(t+T) - x(t) \tag{2}$$

This quantity was introduced by Kolmogorov in studies of the theory of turbulence, and under the name of structure function of x its second moment has seen wide use in meteorology and elsewhere [Lindsay and Chie, 1976]. The attraction of looking at the variation of x over a fixed time T is that $y_T(t)$ is often stationary, and thus easily characterized statistically, even when $x(t)$ is nonstationary (as must be true for $v \leq -1$). But the statistics of y_T may often be as of much interst as those of x; in particular, if we want to decide whether some recent fluctuation in $x(t)$ is consistent with its past behavior it is to the distribution of y_T that we must turn. Unfortunately, if we choose to summarize this past behavior as having a spectrum of the form (1), the fractional Brownian motions of Mandelbrot and Van Ness [1968] turn out to be very inconvenient, since the expression for the power spectrum of such processes is extremely complicated [Graf, 1983; Geweke and Porter-Hudake, 1983]. There is thus no simple relationship between the spectrum of such processes and their variation in the time domain;

There is however a relatively simple method whereby we can relate the spectral level (1) to the distribution of $y_T(t)$, provided that we only aim to find only the second moment, or variance, of y_T (denoted by $\langle y_T^2 \rangle$), which, as noted above, is the structure function. This restriction is of course unavoidable given that our basic description is

the power spectrum. Specifying only the second moment is adequate to define the distribution of y_T if it is Gaussian. This restriction will not usually be seriously violated for real data, but should be kept in mind before inferences about the complete distribution are made from the value of $<y_T^2>$.

To determine $<y_T^2>$ from the spectrum, we observe that $y(t)$ is derived from $x(t)$ by convolution; we can rewrite (2), in the notation of Bracewell [1965], as

$$y(t) = x(t) * \left[\delta(t-T) - \delta(t) \right] \qquad (3)$$

Straightforward application of the results of Fourier theory then shows that the power spectrum of y_T is given by

$$P_y(f) = |G_T(f)|^2 P_x(f)$$

where $G_T(f)$ is the frequency response of the convolution filter, used to produce $y(t)$ from $x(t)$ namely

$$|G_T(f)|^2 = 4\sin^2\pi fT$$

Finally, since the variance of a random process is equal to the integral of its power spectrum we find

$$<y_T^2> = \int_0^\infty P_y(f)df = \int_0^\infty 4P_x(f)\sin^2\pi fT df \qquad (4)$$

Other methods than (2) of forming auxiliary series exist, and some have long been used in (for example) studies of oscillator stability. Rutman [1978] discusses many of these and shows how the transfer-function approach just discussed can be used to derive their behavior for different spectra. (The structure function is one quantity discussed, but because of its limited usefulness for oscillator studies it is not considered in any detail.)

If we now specialize $P_x(f)$ to the form (1) and make the change of variable $u = \pi fT$, we find

$$<y_T^2> = \frac{4P_0}{f_0^\nu} \frac{T^{-(\nu+1)}}{\pi^{\nu+1}} \int_0^\infty u^\nu \sin^2 u \; du = C_\nu \frac{P_0}{f_0^\nu} T^{-(\nu+1)} \qquad (5)$$

This immediately implies that the standard deviation of y_T, $\sigma_T = \left[<y_T^2>\right]^{1/2}$, is proportional to $T^{-(\nu+1)/2}$, or T^H in Mandelbrot's notation; for $\nu = -2$, we find σ_T proportional to $T^{1/2}$, the familiar result for Brownian motion. The definite integral in (5), and thus the coefficient C_ν, can be found in closed form:

$$C_\nu = \frac{-1}{2^{\nu+1}\pi^\nu\Gamma(-\nu)\cos(\nu\pi/2)}$$

which for $\nu = -2$ gives $C_\nu = 2\pi^2$. Figure 1 shows $C_\nu^{1/2}$ over the range $-3 < \nu < -1$, and illustrates how the expression (4) goes to infinity at both limits of this range because of the divergence in the integral in (5). These divergences occur at opposite limits of the integral; put crudely, as ν approaches -3, the low-frequency fluctuations in $x(t)$ become so large that $y(T)$ becomes nonstationary, while as ν approaches -1, the high-frequency fluctuations in $x(t)$ approach an ultraviolet catastrophe, with infinite

variance at high frequencies. This latter divergence will not be a concern in practice, and could be eliminated in the theory by replacing the δ-functions in (3) with finite-width sampling functions.

A major assumption has been passed over in using equation (1) to go from equation (4) to equation (5); namely, that while (4) presupposes $x(t)$ to possess a power spectrum, any process with an apparent spectral index less than −1 must be nonstationary, so that its spectrum does not exist: a contradiction more apparent than real. It is true that (1), with $v < -1$, cannot describe the spectrum of a stationary process. However, if we suppose the $P_x(f)$ to be described by (1) for $f > f_b$, and to be (say) constant at $P_0(f_b/f_0)^v$ for $f < f_b$, $x(t)$ will be a stationary process, the operations leading from (4) to (5) will be valid, and, the result in (5) will be essentially unchanged as long as $T \ll f_b^{-1}$. While introducing such a cutoff frequency is in one sense arbitrary, it must exist for any actual process (see, for example, Keshner [1982] for such a model for 1/f noise). Since the finite span of our observations will always render us incapable of observing it, there seems to be no reason to avoid introducing it to avoid the difficulties into which a too-strict adherence to an ideal mathematical model would otherwise lead us [Slepian, 1978].

3. Comparison with Spectral Crossover Approach

Agnew [1987] described another problem relating to power-law processes: How to compare a record whose errors are of this form with measurements with independent error σ made at regular time intervals Δ. (This characterizes the problem of comparing crustal deformation measurements made using strainmeters and tiltmeters with those made by geodetic methods.) For a spectral index less than −1, there will be some frequency at which the fluctuations in the power-law errors equal the error gotten by averaging the independent measurements. At a higher frequency the power-law errors will be smaller and using records with such errors will give a better result; at lower frequency the independent errors, suitably averaged, will be superior.

This problem is easily solved if cast in spectral form (Figure 2). The spectrum of the independent-error measurements must be constant, with level P_m, from 0 to the Nyquist frequency, $(2\Delta)^{-1}$. Since the integral of the spectrum is the variance, $P_m = 2\sigma^2\Delta$. This equals the power-law spectrum (1) at a crossover frequency

$$f_c = \left[\frac{2\sigma^2 \Delta f_0^v}{P_0} \right]^{1/v} \tag{6}$$

which thus sets the boundary between one or another process having a lower level. The crossover frequency can be equally easily obtained graphically for a spectrum of more general shape, though a closed-form expression becomes cumbersome. To give a concrete example, we may compare the strain spectrum shown in Figure 3 with repeated distance measurements with a σ of 10^{-7}. If these were made weekly, the equivalent spectral level would be 1.2×10^{-8} ϵ^2/Hz, or −79 dB, giving a crossover frequency corresponding to a period of 300 days; if they were made daily, this period becomes 200 days.

The theory developed in Section 2 gives another way of looking at this problem. At a period $T_c = f_c^{-1}$, the rms fluctuation in the power-law process will be (from (5)

and (6))

$$\langle y_{T_c}^2 \rangle^{\frac{1}{2}} = C_v^{\frac{1}{2}} \left[\frac{P_0}{f_0^v} \right]^{-v/2} 2^{(v+1)/(2v)} \sigma^{(1+v)/v} \Delta^{(1+v)/(2v)}$$

The error in the independent-measurement series, suitably averaged, will be $N^{-\frac{1}{2}}\omega$ where N is the number of measurements; obviously for regular sampling $N \approx T_c/\Delta$. Again using (6), we find

$$\frac{\sigma N^{-\frac{1}{2}}}{\langle y_{T_c}^2 \rangle^{\frac{1}{2}}} = \frac{1}{(2C_v)^{\frac{1}{2}}}$$

which is always less than one. This is as it should be, since at the crossover frequency the independent measurements should be capable of resolving the fluctuations of the power-law series.

4. Generalizations

As noted in Section 2, it is straightforward to modify the convolution (3) to take account of a finite-length sampling interval; such a modification eliminates the ultraviolet catastrophe for $v > -1$. A perhaps less obvious generalization allows us to extend this method to (in principle) spectral indices less than -3. The basis for this is the recognition that the convolution function in (3) is equivalent to the fundamental wavelet of Robinson and Treitel [1980] denoted by the sequence $(1, -1)$. The frequency response of this can be shown to have two zeroes at zero frequency; it is these zeroes, of course, which cancel out the singularity in the integral in (5). This immediately suggests that more extreme singularities could be removed by adding more zeroes, such addition can be achieved by convolving fundamental wavelets together. For example, convolving two wavelets gives the sequence $(1, -2, 1)$; the corresponding sampling sequence

$$w(t) = x(t+2T) - 2x(t+T) + x(t)$$

is equivalent to convolution with a function whose gain has four zeroes at $f = 0$. The quantity $\langle w^2 \rangle$ thus will be well defined for $-5 < v < -1$. This approach makes it easy to see how to design higher-order versions of the usual structure function, something that is less clear in the usual treatments of this subject [Lindsay and Chie, 1976].

Another generalization is to note that (4) applies for a general spectral shape $P_x(f)$, provided that at low frequencies $P_x(f)$ increases less rapidly than f^{-3}, and at high frequencies decreases more rapidly than f^{-1}. Of course, we then will not usually be able to find a closed-form expression for $\langle y_T^2 \rangle$, but must calculate it numerically. This allows us to proceed even in the face of the departures from power-law (or fractal) behavior noted, for example, by Gilbert [1989]. A simple spectral shape which fits many spectra quite well is a piecewise power-law form; for $i=1, \ldots$

$$P_x(f) = P_i \left[\frac{f}{f_i} \right]^{v_i} \quad f_{i-1} < f \le f_i \tag{6}$$

with f_0 being set to zero. The integral (4) then becomes the sum of integrals over each frequency interval. Because of the oscillatory nature of the integrand and the wide range in frequency, the integration must be done with some care. If f_1 is set to

a value such that $f_1T \ll 1$, we may approximate $\sin \pi fT$ by πfT to get

$$\int_{f_{i-1}}^{f_i} P_x(f) \sin^2 \pi fT \; df = \frac{P_1 \pi^2 T^2}{(v+3) f_i^{v_i}} (f_i^{v_i+3} - f_{i-1}^{v_i+3}) \tag{7}$$

For the other integrals it is useful to write $\sin^2 \pi fT = \tfrac{1}{2}(1 - \cos 2\pi fT)$, whence

$$\int_{f_{i-1}}^{f_i} P_x(f) \sin^2 \pi fT \; df = \frac{P_i}{2 f_i^v} \left[\left[\frac{f_i^{v+1} - f_{i-1}^{v+1}}{v+1} \right] - \int_{f_{i-1}}^{f_i} f^v \cos 2\pi fT df \right] \tag{8}$$

For T or f sufficiently large, the integral in the second expression will clearly be close to zero. We thus evaluate $<y_T^2>$ in three ways: for $fT \ll 1$, we use (7); for $fT \gg 1$, we use (8) with the integral over cosine omitted; and for intermediate values we use the full expression in (8). For $v > 0$, this means calculating the integral numerically; for $v < 0$, we may use continued-fraction or Taylor-series approximations to the incomplete gamma function.

5. Applications

Figure 3a shows the spectrum of earth strain (northwest-southeast extension) measured at Piñon Flat Observatory in southern California. The peak at high frequencies is caused by microseisms; the narrow peaks at multiples of 1 cycle/day $(1.16 \times 10^{-5}$ Hz) are caused by earth tides and thermal effects. Except for these narrow peaks, which being largely deterministic can be predicted and removed from the data, the spectrum is very well fit by equation (6) the piecewise power-law model. Figure 3b shows the value of $<y_T^2>^{1/2}$ computed from this fit, and also for the case where the spectrum is assumed to fall off as f^{-2} above 0.1 Hz, as would be so if the data were highpassed to remove microseismic energy. At periods from 10 to 100 seconds, Figure 3b shows a constant value of 0.05 $n\varepsilon$, which may be taken to be the resolution limit of this data for such rapid changes as the coseismic offsets discussed by Wyatt [1988]. At longer times the fluctuations increase steadily.

If we had, in Figure 3, plotted rates of rms strain change against time interval, we would see that the longer the time interval, the slower the apparent rate. It has been a frequent observation of geologists that many rates of deformation, examined over long times, appear to be much less than those determined over shorter times (for example, geodetically). But such behavior is exactly what would be expected if the deformations being considered, like those of shorter period shown in Figure 3, had a power-law spectrum with index between −3 and −1. Such a stochastic model allows us to reconcile apparent changes of rate with strict uniformitarianism (the "null" hypothesis of Gilluly [1949]): the present will always appear to be the most active period, whenever it happens to be.

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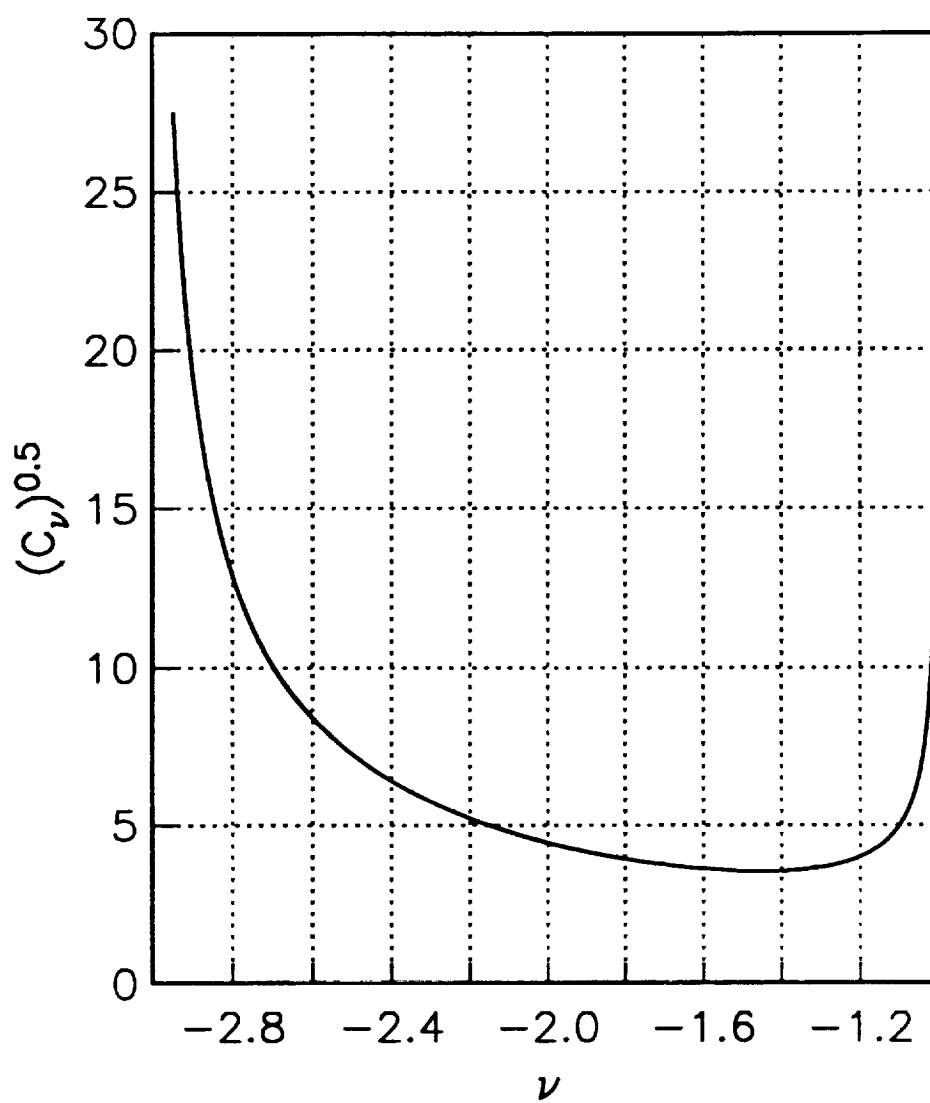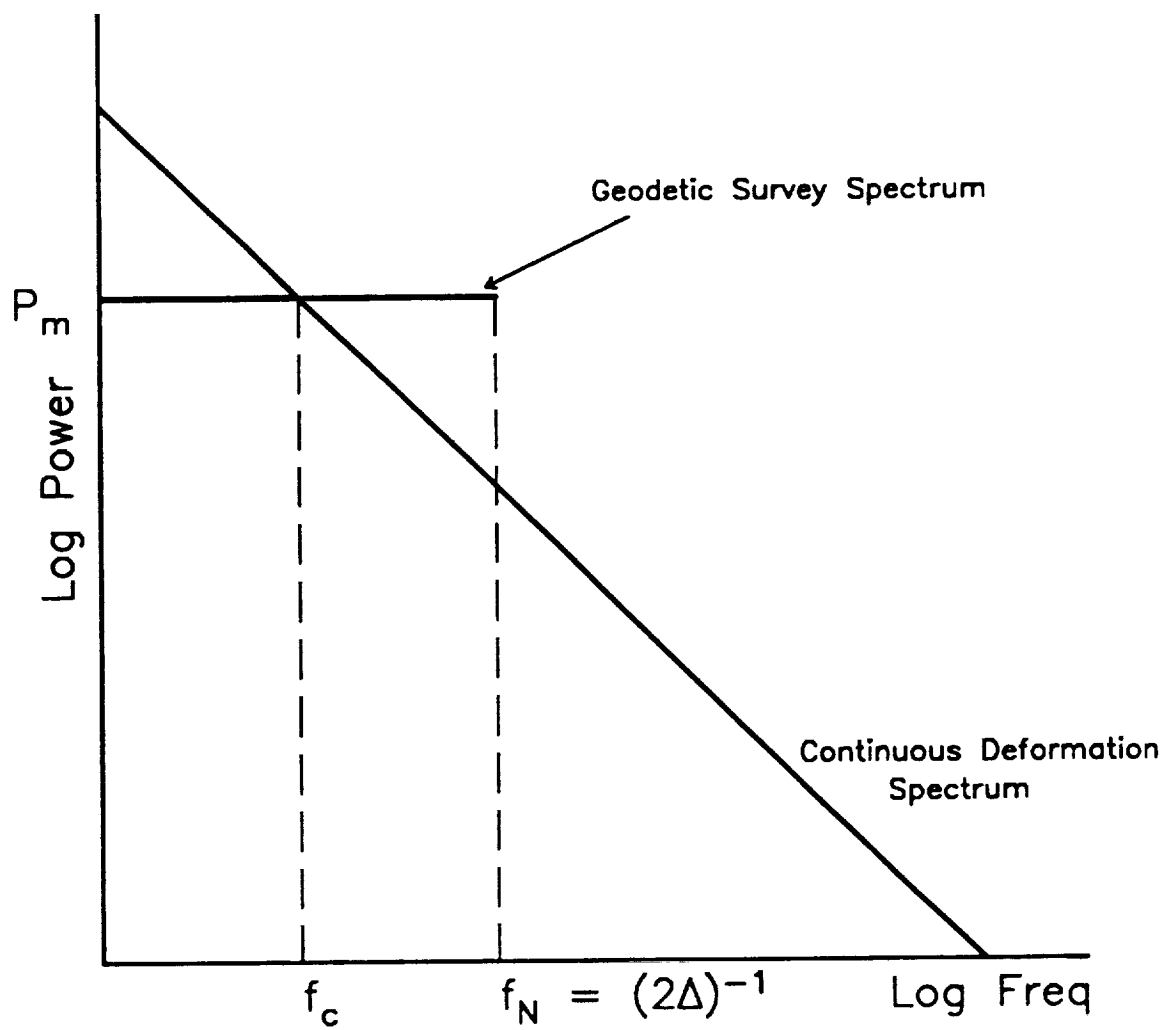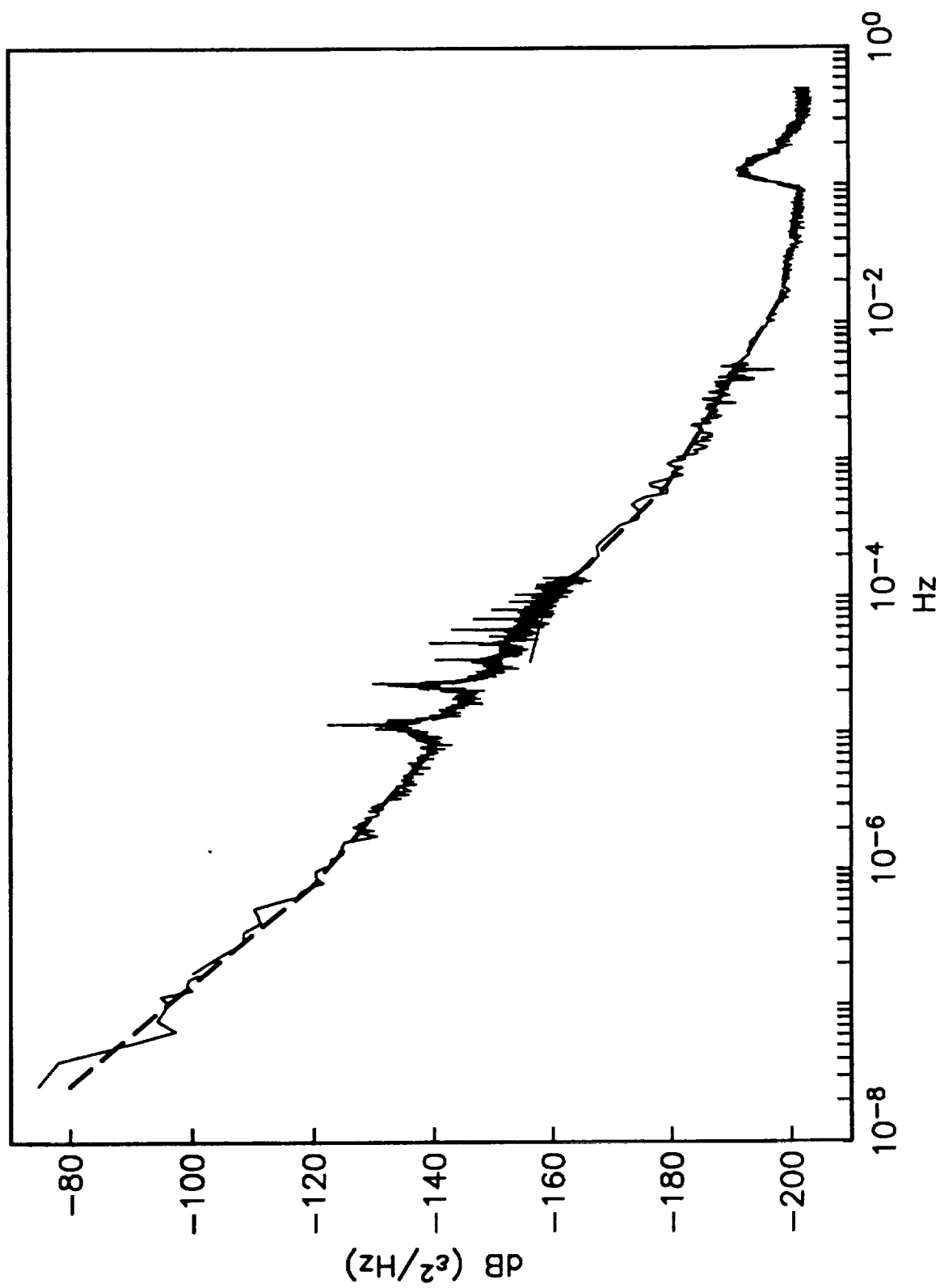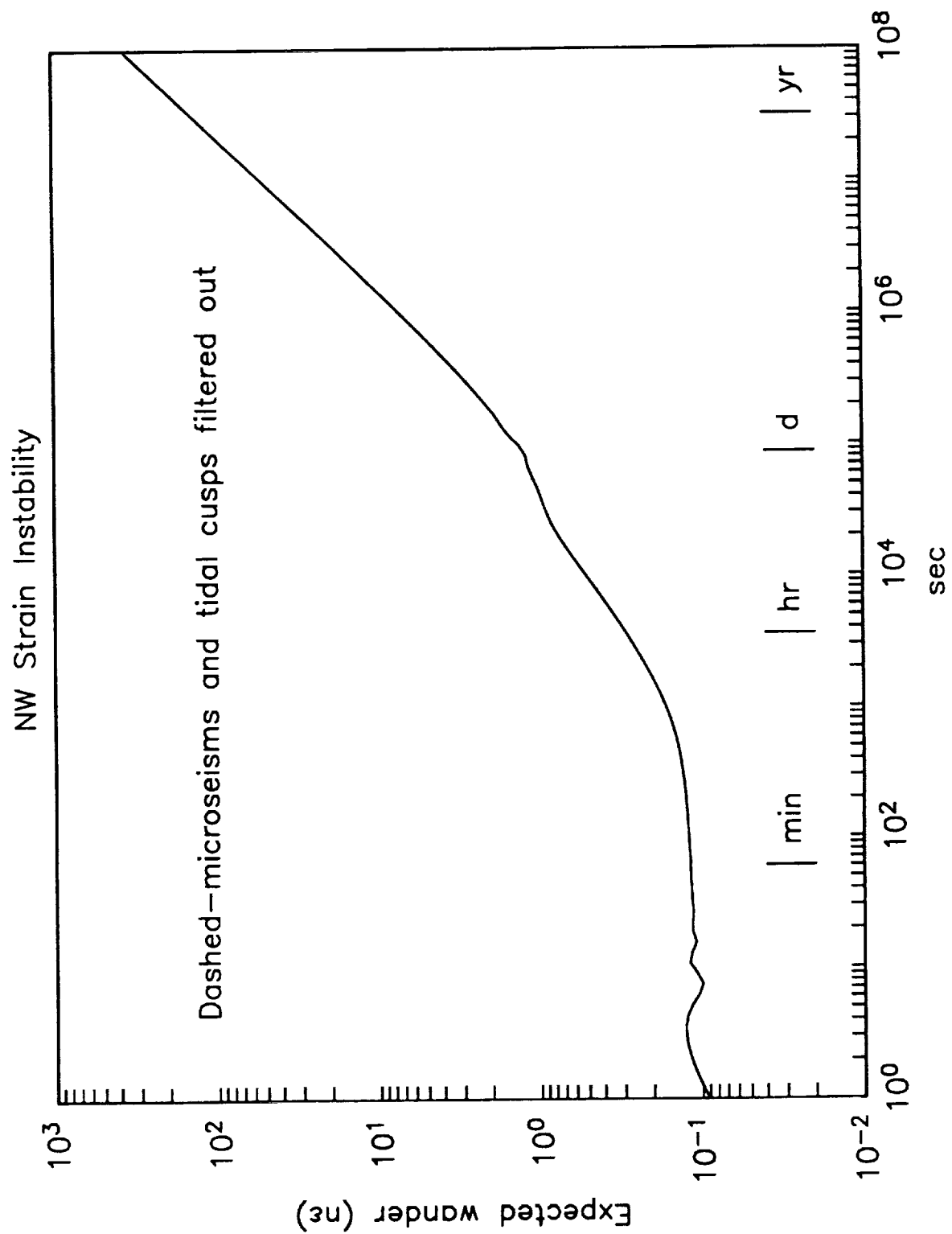Figure 1

Figure 2

NW Strain PSD (1990:31-33)

dB (ε^2/Hz)

Hz

Figure 3a

NW Strain Instability

Dashed—microseisms and tidal cusps filtered out

Expected wander (ne)

sec

| min | hr | d | yr

10^{-2} 10^{-1} 10^0 10^1 10^2 10^3

10^0 10^2 10^4 10^6 10^8

Figure 3b